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# A NEW PROOF FOR A ROLEWICZ'S TYPE THEOREM: AN EVOLUTION SEMIGROUP APPROACH

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ABSTRACT. Let  $\mathbb{R}_+$  be the set of all non-negative real numbers and  $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\}$  be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on a Banach space  $X$ . Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a non-decreasing function such that  $\varphi(t) > 0$  for all  $t > 0$ . We prove that if there exists  $M_\varphi > 0$  such that

$$\sup_{s \geq 0} \int_s^\infty \varphi(\|U(t, s)x\|) dt = M_\varphi < \infty, \text{ for all } x \in X, \|x\| \leq 1,$$

then  $\mathcal{U}$  is uniformly exponentially stable. For  $\varphi$  continuous, this result is due to S. Rolewicz.

## 1. INTRODUCTION

Let  $X$  be a real or complex Banach space and  $L(X)$  the Banach algebra of all linear and bounded operators on  $X$ . Let  $\mathbf{T} = \{T(t) : t \geq 0\} \subset L(X)$  be a strongly continuous semigroup on  $X$  and  $\omega_0(\mathbf{T}) = \lim_{t \rightarrow \infty} \frac{\ln(\|T(t)\|)}{t}$  be its growth bound. The Datko-Pazy theorem ([1], [2]) states that  $\omega_0(\mathbf{T}) < 0$  if and only if for all  $x \in X$  the maps  $t \mapsto \|T(t)x\|$  belongs to  $L^p(\mathbb{R}_+)$  for some  $1 \leq p < \infty$ .

A family  $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\} \subset L(X)$  is called an *evolution family* of bounded linear operators on  $X$  if  $U(t, t) = \mathbf{I}$  (the identity operator on  $X$ ) and  $U(t, \tau)U(\tau, s) = U(t, s)$  for all  $t \geq \tau \geq s \geq 0$ . Such a family is said to be *strongly continuous* if for every  $x \in X$ , the maps

$$(t, s) \mapsto U(t, s)x : \{(t, s) : t \geq s \geq 0\} \rightarrow X$$

are continuous, and *exponentially bounded* if there are  $\omega > 0$  and  $K_\omega > 0$  such that

$$(1.1) \quad \|U(t, s)\| \leq K_\omega e^{\omega(t-s)} \text{ for all } t \geq s \geq 0.$$

The family  $\mathcal{U}$  is called *uniformly exponentially stable* if (1.1) holds for some negative  $\omega$ . If  $\mathbf{T} = \{T(t) : t \geq 0\} \subset L(X)$  is a strongly continuous semigroup on  $X$ , then the family  $\{U(t, s) : t \geq s \geq 0\}$  given by  $U(t, s) = T(t-s)$  is a strongly continuous and exponentially bounded evolution family on  $X$ . Conversely, if  $\mathcal{U}$  is a strongly continuous evolution family on  $X$  and  $U(t, s) = U(t-s, 0)$  then the family  $\mathbf{T} = \{T(t) : t \geq 0\}$  given by  $T(t) = U(t, 0)$  is a strongly continuous semigroup on  $X$ .

The Datko-Pazy theorem can be obtained from the following result given by S. Rolewicz ([3], [4]).

*Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a continuous and nondecreasing function such that  $\varphi(0) = 0$  and  $\varphi(t) > 0$  for all  $t > 0$ . If  $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\} \subset L(X)$  is a strongly*

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continuous and exponentially bounded evolution family on the Banach space  $X$  such that

$$(1.2) \quad \sup_{s \geq 0} \int_s^\infty \varphi(\|U(t, s)x\|) dt = M_\varphi < \infty, \text{ for all } x \in X, \|x\| \leq 1,$$

then  $\mathcal{U}$  is uniformly exponentially stable.

A shorter proof of the Rolewicz theorem was given by Q. Zheng [5] who removed the continuity assumption about  $\varphi$ . Other proofs of (the semigroup case) Rolewicz's theorem were offered by W. Littman [6] and J. van Neervan [7, pp. 81-82]. Some related results have been obtained by K.M. Przyłuski [8], G. Weiss [13] and J. Zabczyk [9].

In this note we prove the following:

**Theorem 1.** *Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a nondecreasing function such that  $\varphi(t) > 0$  for all  $t > 0$ . If  $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\} \subset L(X)$  is a strongly continuous and exponentially bounded evolution family of operators on  $X$  such that (1.2) holds, then  $\mathcal{U}$  is uniformly exponentially stable.*

Our proof of Theorem 1 is very simple. In fact, we apply a result of Neerven (see below) for the evolution semigroup associated to  $\mathcal{U}$  on  $C_{00}(\mathbb{R}_+, X)$ , the space of all continuous,  $X$ -valued functions defined on  $\mathbb{R}_+$  such that  $f(0) = \lim_{t \rightarrow \infty} f(t) = 0$ .

**Lemma 1.** *Let  $\mathcal{U}$  be a strongly continuous and exponentially bounded evolution family of operators on  $X$  such that*

$$(1.3) \quad \sup_{s \geq 0} \int_s^\infty \varphi(\|U(t, s)x\|) dt = M_\varphi(x) < \infty, \text{ for all } x \in X.$$

Then  $\mathcal{U}$  is uniformly bounded, that is,

$$\sup_{t \geq \xi \geq 0} \|U(t, \xi)\| < \infty.$$

*Proof of Lemma 1.* Let  $x \in X$  and  $N(x)$  be a positive integer such that  $M_\varphi(x) < N(x)$  and let  $s \geq 0, t \geq s + N$ . For each  $\tau \in [t - N, t]$ , we have

$$(1.4) \quad \begin{aligned} e^{-\omega N} 1_{[t-N, t]}(u) \|U(t, s)x\| &\leq e^{-\omega(t-\tau)} 1_{[t-N, t]}(u) \|U(t, \tau)U(\tau, s)x\| \\ &\leq K_\omega \|U(u, s)x\|, \end{aligned}$$

for all  $u \geq s$ . Here  $K_\omega$  and  $\omega$  are as in (1.1) and  $\omega > 0$ .

If we choose  $x = 0$  in (1.3), then we get  $\varphi(0) = 0$ , and thus from (1.4) we obtain

$$(1.5) \quad \begin{aligned} N(x) \varphi\left(\frac{\|U(t, s)x\|}{K_\omega e^{\omega N}}\right) &= \int_s^\infty \varphi\left(\frac{1_{[t-N, t]}(u) \|U(t, s)x\|}{K_\omega e^{\omega N}}\right) du \\ &\leq \int_s^\infty \varphi(\|U(u, s)x\|) du \leq M_\varphi(x). \end{aligned}$$

We assume that  $\varphi(1) = 1$  (if not, we replace  $\varphi$  by some multiple of itself). Moreover, we may assume that  $\varphi$  is a strictly increasing map. Indeed if  $\varphi(1) = 1$  and  $a := \int_0^1 \varphi(t) dt$ , then the function given by

$$\bar{\varphi}(t) = \begin{cases} \int_0^t \varphi(u) du, & \text{if } 0 \leq t \leq 1 \\ \frac{at}{at + 1 - a}, & \text{if } t > 1 \end{cases}$$

is strictly increasing and  $\bar{\varphi} \leq \varphi$ . Now  $\varphi$  can be replaced by some multiple of  $\bar{\varphi}$ . From (1.5) it follows that if  $t \geq s + N(x)$  and  $x \in X$ , then

$$\|U(t, s)\| \leq K_\omega e^{\omega N(x)}, \quad \text{for all } x \in X.$$

Using this inequality and the exponential boundedness of the evolution family, we have that

$$(1.6) \quad \sup_{t \geq \xi \geq 0} \|U(t, \xi)x\| \leq K_\omega e^{\omega N(x)}, \quad \text{for each } x \in X.$$

The conclusion of Lemma 1 follows from (1.6) and the Uniform Boundedness Theorem. ■

Let  $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\}$  be a strongly continuous and exponentially bounded evolution family of bounded linear operators on  $X$ . We consider the strongly continuous evolution semigroup associated to  $\mathcal{U}$  on  $C_{00}(\mathbb{R}_+, X)$ . This semigroup is defined by

$$(1.7) \quad (\mathfrak{T}(t)f)(s) := \begin{cases} U(s, s-t)f(s-t), & \text{if } s \geq t \\ 0, & \text{if } 0 \leq s \leq t \end{cases}, \quad t \geq 0$$

for all  $f \in C_{00}(\mathbb{R}_+, X)$ . It is known that  $\mathfrak{T} = \{\mathfrak{T}(t) : t \geq 0\}$  is a strongly continuous semigroup and in addition  $\omega_0(\mathfrak{T}) < 0$  if and only if  $\mathcal{U}$  is uniformly exponentially stable ([10], [11], [12]).

*Proof of Theorem 1.* Let  $\varphi$  be as in Theorem 1. We assume that  $\varphi(1) = 1$ . Then

$$\Phi(t) := \int_0^t \varphi(u) du \leq \varphi(t) \quad \text{for all } t \in [0, 1].$$

Without loss of generality we may assume that

$$\sup_{t \geq 0} \|\mathfrak{T}(t)\| \leq 1,$$

where  $\mathfrak{T}$  is the semigroup defined in (1.7). Then for all  $f \in C_{00}(\mathbb{R}_+, X)$  with  $\|f\|_\infty \leq 1$ , one has

$$\begin{aligned} & \int_0^\infty \Phi(\|\mathfrak{T}(t)f\|_{C_{00}(\mathbb{R}_+, X)}) dt \\ &= \int_0^\infty \Phi\left(\sup_{s \geq t} \|U(s, s-t)f(s-t)\|\right) dt = \int_0^\infty \Phi\left(\sup_{\xi \geq 0} \|U(t+\xi, \xi)f(\xi)\|\right) dt \\ &= \int_0^\infty \left( \int_0^\infty 1_{\left[0, \sup_{\xi \geq 0} \|U(t+\xi, \xi)f(\xi)\|\right]}(u) \varphi(u) du \right) dt \\ &= \sup_{\xi \geq 0} \int_0^\infty \left( \int_0^\infty 1_{[0, \|U(t+\xi, \xi)f(\xi)\|]}(u) \varphi(u) du \right) dt \\ &= \sup_{\xi \geq 0} \int_0^\infty \Phi(\|U(t+\xi, \xi)f(\xi)\|) dt \leq \sup_{\xi \geq 0} \int_0^\infty \varphi(\|U(t+\xi, \xi)f(\xi)\|) dt \\ &= \sup_{\xi \geq 0} \int_\xi^\infty \varphi(\|U(\tau, \xi)f(\xi)\|) d\tau \leq M_\varphi < \infty, \end{aligned}$$

where  $1_{[0, h]}$  denotes the characteristic function of the interval  $[0, h]$ ,  $h > 0$ .

Now, from [7, Theorem 3.2.2], it follows that  $\omega_0(\mathfrak{T}) < 0$ , hence  $\mathcal{U}$  is uniformly exponentially stable. ■

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